to finish our proof we must show that μ^* is a countably additive measure

Lemma. *μ^* is countably additive on $\mathcal{M}_F$*

Proof. Want to show that $B_i \in \mathcal{M}_F$, $B_i \cap B_j = \emptyset$, $i \neq j$ $\bigcup_{i=1}^{\infty} B_i = B \in \mathcal{M}_F$, then $\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B_i)$

We can easily prove inequality one way.

$$B_N = \bigcup_{i=1}^{N} B_i \Longrightarrow B_N \subset B$$

and we get the following strings of inequalities

$$\mu^*(B) \geq \mu^*(B_N) = \sum_{i=1}^{N} \mu^*(B_i) \Longrightarrow \mu^*(B) \geq \sum_{i=1}^{N} \mu^*(B_i)$$

now we need to show countably sub-additiviy, that is $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$. Given $\epsilon > 0$, $\forall i$, $\exists \{A_{i,j}\}_{j=1}^{\infty} \subset \mathcal{R}$ a cover of A_i such that

$$\mu^*(A_i) + \frac{\epsilon}{2^i} \geq \sum_{j=1}^{\infty} \mu^*(A_{i,j})$$

the collection $\{A_{i,j}\}_{i,j=1}^{\infty}$ covers $\bigcup_{i=1}^{\infty} A_i$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i,j=1}^{\infty} \mu^*(A_{i,j}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu^*(A_{i,j}) \leq \sum_{i=1}^{\infty}\left[\mu^*(A_i) + \frac{\epsilon}{2^i}\right] = \epsilon + \sum_{i=1}^{\infty} \mu^*(A_i)$$

this is true $\forall \epsilon$, so $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, and so μ^* is countably additive on $\mathcal{M}_F$ $\square$

Lemma. $\mathcal{M}_F = \{A \in \mathcal{M} | \mu^*(A) < \infty\}$

Proof. We would like to prove that if $\mu^*(A) < \infty$ and $A \in \mathcal{M}$ then $A \in \mathcal{M}_F$. Since $A \in \mathcal{M}$ we know that $A = \bigcup_{j=1}^{\infty} B_j, B_j \in \mathcal{M}_F$. We can replace the B_j's by a disjoint sequence in $\mathcal{M}_F$, but with the same measure.

$$A = \bigcup_{j=1}^{\infty} B'_j, \qquad B'_N = B_N \setminus \bigcup_{i=1}^{N-1} B_i \in \mathcal{M}_F, B_i \text{ disjoint}$$

Then

$$\bigcup_{j=1}^{N} B'_j \subset A \Longrightarrow \mu^*\left(\bigcup_{j=1}^{N} B'_j\right) \leq \mu^*(A) < \infty, \qquad \forall N$$

and we have that and so

$$\mu^*\left(\bigcup_{j=1}^{N} B'_j\right) = \sum_{i=1}^{N} \mu^*(B'_j) \leq \mu^*(A)$$

let $C_N = \bigcup_{j=1}^N B'_j$ then $C_N \in \mathcal{M}_F$ and so

$$\mu^*(A \ominus C_N) = \mu^*\left(\bigcup_{j>N} B'_j\right) \le \sum_{j>N} \mu^*(B'_j) < \epsilon$$

for any ϵ for N large enough. And so $\mu^*(A \ominus B_N) \to 0$, $A \in \mathcal{M}_F$ since $\mathcal{M}_F$ is closed under $\mu^*(X \ominus X)$. $\square$

Definition. A collection of subsets $\mathcal{N} \subset 2^X$ is a σ-ring if

1. $\mathcal{N}$ is a ring
2. If $A_i \in \mathcal{N}$ then $\bigcup_{i=1}^\infty A_i \in \mathcal{N}$.[closure under countable unions]

Theorem. *$\mathcal{M}$ is a σ-ring.*

Proof. Suppose $A_i \in \mathcal{M}$ we need to show that $\bigcup_{i=1}^\infty A_i \in \mathcal{M}$. Well,

$$A_i = \bigcup_{i=1}^\infty B_{ij}, B_{ij} \in \mathcal{M}_F \Longrightarrow \bigcup_{i=1}^\infty A_i = \bigcup_{i,j=1}^\infty B_{ij}$$

countable union of countable unions is still countable, so this is indeed in $\mathcal{M}$.

We also need to prove that its a ring. We have already proved the condition for unions. So we have to prove that $A \ominus B \in \mathcal{M}$ if $A, B \in \mathcal{M}$, it suffices to show that $A \setminus B \in \mathcal{M}$, since we already know closure under unions. Now,

$$A \setminus B = \left(\bigcup_{i=1}^\infty A_i\right) \cap \left(\bigcup_{i=1}^\infty B_i\right)^c = \bigcup_{i=1}^\infty A_i \cap \bigcap_{i=1}^\infty B_i^c = \bigcup_{i=1}^\infty (A_i \setminus B), \qquad B = \bigcap_{i=1}^\infty B_i^c$$

$A_i \setminus B$ is in $\mathcal{M}_F$ (Melrose says think about it) $\square$

Theorem. *If $A_i \in \mathcal{M}$ and $A_i \cap A_j = \emptyset, i \neq j$ then*

$$\mu^*\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu^*(A_i)$$

i.e. μ^* is a measure on $\mathcal{M}$

Proof. Let $A = \bigcup_{i=1}^\infty A_i$ there are two cases

- $\mu^*(A) = \infty$. Then

  $$\mu^*(A) \le \sum_{i=1}^\infty \mu^*(A_i)$$

  so $\sum \mu^*(A_i) = \infty$

- $\mu^*(A) \neq \infty$ then $A \in \mathcal{M}_F$ and $\bigcup_{i=1}^N A_i \subset A$ and so

$$\mu^*\left(\bigcup_{i=1}^N A_i\right) \leq \mu^*(A) \Longrightarrow \bigcup_{i=1}^N A_i \in \mathcal{M}_F$$

□

So we have extended μ from $\mathcal{R}$ to $\mathcal{M}_F$ to $\mathcal{M}$.

1.3 Extending Lebesgue Measure

Lebesgue measures $(X, \mathcal{R}, \mu)$ is countably additive and we extended it to $(X, \mathcal{M}, \mu)$ where $\mathcal{M}$ is a σ-ring. Apply these constructions to $X = \mathbb{R}^n$, $\mathcal{R} = \mathcal{R}_{\text{Leb}}$ (disjoint union of a finite number of rectangles). So we can generate $(\mathbb{R}^n, \mathcal{M}, \mu)$. $\mathcal{M}$ is closed under countable unions. $E \subset \mathcal{M}$ then $E = \bigcup_{i=1}^\infty E_i, E_i \in \mathcal{M}_F$. And $F \in \mathcal{M}_F$, there exist $A_i \in \mathcal{R}_{\text{Leb}}$ such that $\mu^*(E \ominus A) \to 0$.

Theorem. *Every open set in $\mathbb{R}^n$ is in $\mathcal{M}$ (i.e. is Lebesgue measurable)*

Proof. Consider elements of $\mathbb{R}^n$ with rational endpoints, $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Consider $U \subset \mathbb{R}^n$, open. And consider all multisets of the type above in U. Then

$$U = \bigcup_{R \subset U} R, \qquad R \text{ the rational sets}$$

So U is open and measurable. $\mathbb{R}^n$ is itself measurable, so closed sets are in $\mathcal{M}$, since for E open $\mathcal{R}^n \setminus E \in \mathcal{M}$ is open and so $E^c \in \mathcal{M}$. □

Recall that $\mathcal{M}$ is a σ-ring, closed under countable unions and intersections. $\mathcal{F}_1, \mathcal{F}_2$ are σ-ring subsets of $\mathbb{R}^n$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-ring.

Definition. The Borel sets of $\mathbb{R}^n$ are the elements of the smallest σ-ring containing all open (and closed) sets of $\mathbb{R}^n$.

There *is* a smallest σ-ring because we can take intersections of all such σ-rings, and get one contained in all of them.

Theorem. *If $A \in \mathcal{M}_{Leb}$, then there exists $B \subset \mathcal{B}$ (the borel ring) such that $B \subset A$ and $\mu^*(A \setminus B) = 0$*

Proof. Do this in three steps:

1. If $A \in \mathcal{M}_{\text{Leb}}$, then there exists a borel set $G \subset \mathcal{B}$, such that $G \supset A$ and $\mu^*(G \setminus A) < \epsilon$ (this is not quite what we want, since G contains A)

   NB $\mathcal{R}_{\text{Leb}} \subset \mathcal{B}$, so if $A \in \mathcal{M}_F$, then $\exists A_i \in \mathcal{R}_{\text{Leb}}$ (A_i's are multi-intervals), $A \subset \bigcup_{i=1}^\infty A_i$ implies that $\mu(A) + \epsilon > \sum^\infty \mu(A_i)$, so $B = \bigcup_{i=1}^\infty A_i \in \mathcal{B}$, then $\mu(A \setminus B) < \epsilon$ (so if the measure is finite the definition automatically gives the theorem).

In general $A = \bigcup_{i=1}^{\infty} E_i$, $E_i \in \mathcal{M}_F$. Apply the previous step to each E_i find $B_i \in \mathcal{B}$, $B_i \supset E_i$ such that $\mu^*(E_i \setminus B_i) < \epsilon/2^i$. and then $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$, $\mu^*(A \setminus B) = \sum_{i=1}^{\infty} \mu^*(E_i \setminus B_i) < \epsilon$.

2. If $A \in \mathcal{M}$, $\exists B \in \mathcal{B}$, $B \subset A$ with $\mu(A \setminus B) < \epsilon$. Apply (1) to $\mathbb{R}^n \setminus A$ then we get $E \subset \mathcal{B}$ such that $E \supset \mathbb{R}^n \setminus A$, $\mu^*(E \setminus (\mathbb{R}^n \setminus A)) < \epsilon$, but if we set $B = \mathbb{R}^n \setminus E$ then $B \subset A$ and $\mu^*(A \setminus B) < \epsilon$.

3. Finally, use (2) to construct $F_N \in \mathcal{B}$, $F_N \subset A$, such that $\mu^*(A \setminus F_N) < 1/N$, then $B = \bigcup_{N=1}^{\infty} F_N \in \mathcal{B}$, $B \subset A$ and $\mu^*(A \backslash B)$, $A \backslash B \subset A \backslash F_N$,$\mu^*(A \backslash B) \leq \mu^*(A \backslash F_N) < 1/N$, so $\mu^*(A \setminus B) = 0$. We are done

$\square$

so the Lebesgue sets are "trapped" between Borel sets.

MEASURE THEORY IS DONE.